Journal of Engineering Mathematics **28**: 407–425, 1994.

Further results concerning the refined theory of anisotropic laminated composite plates

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Received 24 May 1993; accepted in revised form 7 December 1993

Abstract. A simple refined discrete-layer theory of anisotropic laminated composite plates is substantiated. The theory is based on the assumption of a piecewise linear variation of the in-plane displacement components and of the constancy of the transverse displacement throughout the thickness of the laminate. This plate model incorporates transverse shear deformation, dynamic and thermal effects as well as the geometrical non-linearities and fulfills the continuity conditions for the displacement components and transverse shear stresses at the interfaces between laminae. As it is shown in the paper, the refinement implying the fulfillment of continuity conditions is not accompanied by an increase of the number of independent unknown functions, as implied in the standard first order transverse shear deformation theory. It is also shown that the within the framework of the linearized static counterpart of the theory, several theorems analogous to the ones in the 3-D elasticity theory could be established. These concern the energetic theorems, Betti's reciprocity theorem, the uniqueness theorem for the solutions of boundary-value problems of elastic composite plates, etc. Finally, comparative remarks on the present and standard first order transverse shear deformation theories are made and pertinent conclusions about its usefulness and further developments are outlined.

1. Introduction

The increased use of advanced composite material systems in various fields of modern technology has stimulated a great deal of interest for the modelling of multilayered composite structures. This interest is caused by the increased amount of publications dealing with the substantiation of refined theories of anisotropic composite plates and shells and by the appearance of several comprehensive survey-papers in the literature [1–4] analyzing in depth the state-of-the-art of the problem.

As it was clearly emphasized: (i) the low stiffness rigidities in transverse shear characterizing these advanced composite material structures, (ii) the drastic variations of transverse shear moduli from layer to layer, (iii) the various damage modes which are susceptible to appear in anisotropic laminated composite structures and (iv) the wide use of non-thin walled composite structures constitute strong arguments towards the implementation and use of refined plate (or shell) models. Such refined structural models have to incorporate transverse shear deformation as well as other effects deciding upon the reliable prediction of the response characteristics. In this sense the extended and refined versions [5–9] of the originally Reissner–Bollé–Mindlin and Yang–Norris–Stavsky first order shear deformation plate theories (FSDT) represent promising avenues towards the fulfillment of this goal.

Other refinements of the theory of laminated composite plates (and shells) have been obtained within the discrete-layer model based on a piecewise linear representation of the in-plane displacement field through the thickness (see e.g. [2, 10–14]). Although the numerical results reveal (see [4, 9, 14, 15]) that the global response characteristics as predicted by this model are very accurate, the number of unknowns and, as a result, the computational complexities, increase greatly with the increase of the number of constituent layers. In addition to this shortcoming, the requirement of continuity of transverse shear stresses at the layer interfaces is violated. Within the discrete-layer model, the continuity requirements of transverse shear stresses at the interlaminar interfaces have been fulfilled in [16–24] for plates and shells and in [25] for beam-type structures.

Such a model is appropriate towards the prediction, for example, of the impact damage or the failure characteristics of postbuckled (mechanical and thermal) composite laminates. In these instances, both the inter- and the intra-lamina response characteristics of composite flat panels have to be accurately determined.

In the majority of cases the assessment of the degree of accuracy of the developed theoretical models is achieved by comparing numerically the obtained response characteristics with their counterparts based (when available) on the 3-D elasticity theory.

Another way of assessing the validity of the 2-D plate (or shell) model is to make evident that, within the respective model, analogous theorems to the ones featured by the 3-D elasticity theory could be established. In addition to the validation of the 2-D structural model, such theorems (e.g. the energetic or the reciprocal ones) have an intrinsic value providing a tool towards the solution of practical problems of the theory of plates (and shells) in general and of the composite ones in particular. It should also be mentioned that by using the former way of validation, a number of comparisons with the available exact numerical results based on the 3-D elasticity theory were done in [19], and the excellent performances of this model were emphasized.

The present paper attempts: i) to generalize and bring new elements in the refined theory of laminated composite anisotropic plates within the assumptions prompted in [18–20] and ii) to establish, within the developed linearized plate theory, analogous theorems to the ones existing in the 3-D elasticity theory. Such a theory characterized by the same number of unknown functions as the FSDT but exhibiting (due to the fulfillment of the interlaminae static continuity conditions) a higher order system of governing equations (i.e., a twelfth order one), may be viewed as a generalization of the standard FSDT. The theory also incorporates the geometrical non-linearities considered in the sense of the von Kármán small strain and moderate rotation concept, as well as the dynamic effects.

Preliminaries

Consider a composite laminated plate consisting of a finite number of linearly elastic anisotropic layers, each of them exhibiting individual physico-mechanical properties. It is assumed that the layers are in perfect bond so that no slip between the adjacent laminae may occur. The (constant) thickness of the kth lamina is denoted by $h_{(k)}$ ($k = 1, \ldots, N$), the total plate thickness is denoted by h while N is the total number of layers.

For the sake of convenience we choose the underformed mid-plane of the bottom layer as the reference plane σ (see Fig. 1). Let x^i, $i = 1, 2, 3$ be the coordinate system to which the points of the plate structure will be referred. It consists of the set of curvilinear in-plane coordinates x^α, $\alpha = 1, 2$, on σ, and the coordinate x^3 normal to σ. The distance (along x^3) between the reference plane and the underformed mid-plane of the kth layer is denoted by ${}^{(k)}Z$, with ${}^{(1)}Z \equiv 0$, while ${}^{(k)}Z^+$ and ${}^{(k)}Z^-$ denote the upper and bottom faces of the kth layer, respectively (see Fig. 1). Let τ be the volume of the plate in the underformed

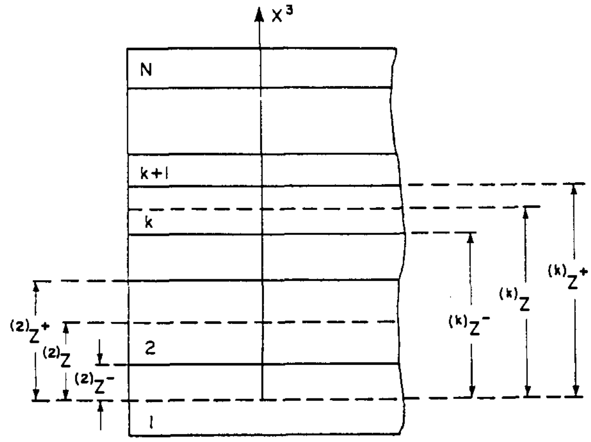

Fig. 1. Geometry of the laminated composite plate.

(reference) configuration. By S^+ and S^- we denote the upper and bottom planes of the plate. Let B denote the lateral boundary surface of τ generated by the normals to σ along its boundary curve C (with arc length s). By B_f and $B_v(B = B_f \cup B_v,\ B_f \cap B_v = \emptyset)$ we denote the two parts of B, where stresses and displacements, respectively, are prescribed.

The components of the metric tensor of the undeformed reference plane σ are

$$\begin{aligned} a_{\alpha\beta} &= \mathbf{a}_\alpha \cdot \mathbf{a}_\beta\,, \quad & a_{\alpha 3} &= \mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0\,, \quad & a_{33} &= \mathbf{a}_3 \cdot \mathbf{a}_3 = 1\,, \\ a^{\alpha\beta} &= \mathbf{a}^\alpha \cdot \mathbf{a}^\beta\,, \quad & a^{\alpha 3} &= \mathbf{a}^\alpha \cdot \mathbf{a}^3 = 0\,, \quad & a^{33} &= \mathbf{a}^3 \cdot \mathbf{a}^3 = 1\,, \end{aligned} \tag{1}$$

where $\mathbf{a}_i$ and $\mathbf{a}^i$, respectively, denote the covariant and contravariant base vectors of σ with $\mathbf{a}_3 = \mathbf{a}^3 \equiv \mathbf{n}$. For the base vectors and metric tensor components in the undeformed 3-D space we have

$$\mathbf{g}_i \equiv \mathbf{a}_i\,, \qquad \mathbf{g}^i \equiv \mathbf{a}^i\,, \qquad g_{ij} \equiv a_{ij}\,, \qquad g^{ij} \equiv a^{ij}\,. \tag{2}$$

Associated with the boundary curve C of σ we define the unit tangent and outward normal vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$, respectively, by

$$\boldsymbol{\tau} = \tau^\alpha \mathbf{a}_\alpha\,; \qquad \boldsymbol{\nu} = \nu^\alpha \mathbf{a}_\alpha = \boldsymbol{\tau} \times \mathbf{n}\,. \tag{3}$$

Partial differentiation will be denoted by a comma, $(\ldots)_{,i} \equiv \partial(\ldots)/\partial x^i$, while the notation $(\cdot\cdot)|_\alpha$ stands for covariant differentiation with respect to x^α. It is mentioned that, for the case of flat plates, by virtue of equations (1) and (2), there is no distinction between space and surface covariant differentiations in τ and on σ, respectively. Throughout the paper the Einsteinian summation convention will be used, with Latin indices ranging from 1 to 3 and Greek indices ranging from 1 to 2. Superscript (k) placed in brackets on the right (or left) of any quantity identifies its affiliation to the kth layer.

Geometric equations

Consider the displacement vector $\mathbf{V}(x^\omega, x^3, t)$ of the 3-D points of the composite plate expressed as:

$$\mathbf{V} = V_\alpha \mathbf{a}^\alpha + V_3 \mathbf{a}^3 , \tag{4}$$

where $V_\alpha \equiv V_\alpha(x^\omega, x^3, t)$ and $V_3 \equiv V_3(x^\omega, x^3, t)$. In order to fulfill the static continuity conditions (associated with the transverse shear stresses) and, at the same time, to obtain the simplest as possible system of governing equations, the following representation for the displacement components is postulated [18–20]:

$$^{(k)}V_\alpha(x^\omega, x^3, t) = v_\alpha + x^3\psi_\alpha + \sum_{l=1}^{N-1} [x^3 - {}^{(l)}Z^+]{}^{(l)}\Omega_\alpha Y(x^3 - {}^{(l)}Z^+) , \tag{5a}$$

$$^{(k)}V_3(x^\omega, x^3, t) = v_3 . \tag{5b}$$

In equations (5) $v_\alpha(\equiv v_\alpha(x^\omega, t))$ and $v_3(\equiv v_3(x^\omega, t))$ denote the displacements of a point of the reference plane of the plate (defined by $x^3 = 0$); ψ_α $(\equiv\psi_\alpha(x^\omega, t))$ denote the rotations of the normal to the reference plane; ${}^{(l)}\Omega_\alpha(\equiv {}^{(l)}\Omega_\alpha(x^\omega, t))$ are functions which have to be determined from the continuity conditions of the transverse shear stresses, i.e.,

$$S^{\alpha 3}|_{x^3 = {}^{(k)}Z^+} = S^{\alpha 3}|_{x^3 = {}^{(k+1)}Z^-} , \tag{6}$$

where S^{ij} denotes the second Piola–Kirchhoff stress tensor, while $Y(\cdot)$ denotes the Heaviside step distribution. It may be remarked that the displacement components ${}^{(k)}V_\alpha$ and ${}^{(k)}V_3$ are continuous functions of x^3 irrespective of the values of ${}^{(l)}\Omega_\alpha$.

Consistent with the representation of displacement components, equation (5), and in conjunction with the Lagrangian strain-displacement relationship used in the spirit of von Kármán partially non-linear theory [26]

$$^{(k)}E_{ij} = \frac{1}{2}({}^{(k)}V_{i|j} + {}^{(k)}V_{j|i} + {}^{(k)}V_{3|i}{}^{(k)}V_{3|j}) , \tag{7}$$

the non-vanishing strain components result as:

$$^{(k)}E_{\alpha\beta} = {}^{(k)}\overset{0}{e}_{\alpha\beta} + x^3\,{}^{(k)}\overset{1}{e}_{\alpha\beta} , \tag{8a}$$

$$^{(k)}E_{\alpha 3} = {}^{(k)}\overset{0}{e}_{\alpha 3} , \tag{8b}$$

where $\overset{m}{e}_{ij} \equiv \overset{m}{e}_{ij}(x^\omega, t)$.

In equation (8) the 2-D strain measures $\overset{m}{e}_{ij}$ are expressed as:

$$2\,{}^{(k)}\overset{0}{e}_{\alpha\beta} = v_{\alpha|\beta} + v_{\beta|\alpha} + v_{3|\alpha}v_{3|\beta} - \sum_{l=1}^{k-1} {}^{(l)}Z^+({}^{(l)}\Omega_{\alpha|\beta} + {}^{(l)}\Omega_{\beta|\alpha}) , \tag{9a}$$

$$2\,{}^{(k)}\overset{1}{e}_{\alpha\beta} = \psi_{\alpha|\beta} + \psi_{\beta|\alpha} + \sum_{l=1}^{k-1} ({}^{(l)}\Omega_{\alpha|\beta} + {}^{(l)}\Omega_{\beta|\alpha}) , \tag{9b}$$

$$2\,{}^{(k)}\overset{0}{e}_{\alpha 3} = \psi_\alpha + v_{3|\alpha} + \sum_{l=1}^{k-1} {}^{(l)}\Omega_\alpha . \tag{9c}$$

It should be remarked that both the representation of the displacement field (5) and the obtained strain measures (9) could be viewed as the superposition of a part which is similar to the one characterizing the standard FSDT and of another part exhibiting a piecewise variation from layer to layer. This latter part is associated with the functions ${}^{(l)}\Omega_\alpha$.

Using the conventional linear relation between the second Piola–Kirchhoff stress and Lagrangian strain tensor components [27], it may be shown that fulfillment of the condition (6) yields for ${}^{(k)}\Omega_\alpha$ the expression:

$$ {}^{(k)}\Omega_\alpha = {}^{(k)}\gamma^{\omega}_{\cdot\alpha}(v_{3|\omega} + \psi_\omega)\,, \tag{10}$$

where

$$ {}^{(k)}\gamma^{\omega}_{\cdot\alpha} \equiv 4\,{}^{(1)}E^{\beta 3\omega 3}({}^{(k+1)}F_{\beta 3\alpha 3} - {}^{(k)}F_{\beta 3\alpha 3})\,. \tag{11}$$

Here and in the following developments E^{ijmn} and F_{ijmn} denote the tensors of elastic moduli and its compliance counterpart, respectively, associated with a medium exhibiting elastic symmetry with respect to $x^3 = 0$. Equations (10) and (11) reveal that:

(i) The transverse shear elastic constants are involved, via equations (10) and (11), in the in-plane displacement quantities and the 2-D strain measures as expressed by equations (5a) and (9), respectively.

(ii) There is a need to apply the present theory whenever the constituent material layers exhibit drastic variations in their transverse shear mechanical properties. Otherwise, if the jump of transverse shear moduli from layer to layer is mild enough, the standard FSDT could be applied. As a result, ${}^{(k)}\gamma^{\lambda}_{\cdot\alpha}$ given by equation (11) could play the role of a test quantity. Depending on its magnitudes, a decision could be made as to whether or not the standard FSDT or its present refined counterpart is advisable to be applied.

(iii) Under the Kirchhoff's hypothesis (implying $\psi_\alpha = -v_{3|\alpha}$) it results ${}^{(k)}\Omega_\alpha \equiv 0$. This entails the conclusion that within the classical plate theory, the refinement brought by the fulfillment of transverse shear stress continuity is redundant.

Equations (5), (9), (10) and (11) show that the present theory is described in terms of the same unknown functions v_α, v_3 and ψ_α as the ones characterizing the usual first order transverse shear deformation theory (FSDT) of composite laminated panels. This fact certainly constitutes an important advantage over those refined plate theories where the number of unknowns is dependent upon the number of constituent layers.

Equations of motion and boundary conditions

In order to derive the equations of motion and boundary conditions of composite laminated plates, Hamilton's variational principle of the 3-D elasticity theory will be used. It is expressed in the form [26]:

$$ \int_{t_0}^{t_1} \mathrm{d}t\left[\int_\tau S^{ij}\,\delta E_{ij}\,\mathrm{d}\tau - \delta K - \int_{A^-} \underset{\sim}{S}^i\,\delta V_i\,\mathrm{d}A - \int_\tau \rho_0 H^i\,\delta V_i\,\mathrm{d}\tau\right] = 0\,. \tag{12}$$

In equation (12), K denotes the kinetic energy of the body; ρ_0 denotes the mass density of the layer materials; $\mathbf{H}(\equiv H^i\mathbf{a}_i)$ denotes the vector of body forces per unit mass of undeformed body; $\underset{\sim}{S}^i$ denotes the components of the stress vector prescribed over the undeformed external boundary A, t_0 and t_1 denote two arbitrary instances of time, where, according to Hamilton's principle, the end conditions are:

$$\delta V_i = 0 \quad \text{for} \quad t = t_0 \quad \text{and} \quad t = t_1 .$$

In equation (12) we express

$$\delta E_{ij} = \frac{1}{2}(\delta V_{i|j} + \delta V_{j|i} + V_{3|i}\,\delta V_{3|j} + V_{3|j}\,\delta V_{3|i}) , \tag{13a}$$

$$\int_0^1 \delta K \,\mathrm{d}t = -\int_{t_0}^{t_1} \mathrm{d}t \int_\tau \rho_0 \ddot{V}^i \,\delta V_i \,\mathrm{d}\tau . \tag{13b}$$

In addition, equations (5) and (6) are used and Green's theorem is applied whenever possible. Considering the variations δv_α, δv_3 and $\delta\psi_\alpha$ as independent and arbitrary, by virtue of the stationary character of the functional (which concerns each instant belonging to the interval $[t_0, t_1]$), the following results are obtained:

(a) The equations of motion

The equations of motion take the following form:

$$\delta v_\alpha: \quad \overset{0}{L}{}^{\alpha\beta}|_\beta + \overset{0}{p}{}^\alpha + \overset{0}{F}{}^\alpha - \overset{0}{I}{}^\alpha = 0 , \tag{14a}$$

$$\delta v_3: \quad Q^\beta|_\beta + (\overset{0}{L}{}^{\alpha\beta} v_{3|\alpha})_{|\beta} + \overset{0}{p}{}^3 + \overset{0}{F}{}^3 - \overset{0}{I}{}^3 = 0 , \tag{14b}$$

$$\delta\psi_\alpha: \quad \overset{1}{L}{}^{\alpha\beta}|_\beta - Q^\alpha + \overset{1}{p}{}^\alpha + \overset{1}{F}{}^\alpha - \overset{1}{I}{}^\alpha = 0 . \tag{14c}$$

In equations (14), the gross stress resultants and stress couples are defined by:

$$\overset{n}{L}{}^{\alpha\beta} = \sum_{k=1}^{N} {}^{(k)}\overset{n}{L}{}^{\alpha\beta} , \quad n = 0, 1 \tag{15a}$$

$$Q^\beta = \sum_{k=1}^{N} \left\{ {}^{(k)}\overset{0}{L}{}^{\beta 3} + \sum_{l=1}^{k-1} [{}^{(k)}\overset{0}{L}{}^{\lambda 3} + {}^{(l)}Z^{+\,(k)}\overset{0}{L}{}^{\lambda}{}_{\gamma} - {}^{(k)}\overset{1}{L}{}^{\lambda\gamma}]\Big|_\gamma \, {}^{(l)}\gamma^{\beta}_{\cdot\lambda} \right\} , \tag{15b}$$

where

$$\{{}^{(k)}\overset{0}{L}{}^{\alpha\beta}, {}^{(k)}\overset{1}{L}{}^{\alpha\beta}\} = \int_{h_{(k)}} {}^{(k)}S^{\alpha\beta}(1, x^3)\,\mathrm{d}x^3 , \tag{16a}$$

$${}^{(k)}\overset{0}{L}{}^{\alpha 3} = \int_{h_{(k)}} {}^{(k)}S^{\alpha 3}\,\mathrm{d}x^3 . \tag{16b}$$

Here the integration is being performed over the thickness of each kth layer implying

$$\int_{h_{(k)}} (\cdot)\,\mathrm{d}x^3 \equiv \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} (\cdot)\,\mathrm{d}x^3 .$$

In connection with the equations of motion (14), an alternative derivation based on a *vectorial approach* was suggested in [28].

(b) The inertia terms

The inertia terms occurring in equations (14) are given by:

$$\overset{0}{I}{}^{\alpha}=\sum_{r=0}^{1}\overset{r}{\ddot{W}}{}^{\alpha}\sum_{k=1}^{N}{}^{(k)}\overset{r}{M}\,, \tag{17a}$$

$$\overset{0}{I}{}^{3}=\ddot{v}_3\sum_{k=1}^{N}{}^{(k)}\overset{0}{M}-\sum_{r=0}^{1}\overset{r}{\ddot{W}}{}^{\lambda}\Big|_{\beta}\sum_{k=1}^{N}\sum_{l=1}^{k-1}[{}^{(k)}\overset{r+1}{M}-{}^{(l)}Z^{+(k)}\overset{r}{M}]^{(l)}\gamma^{\beta}_{\cdot\lambda}\,, \tag{17b}$$

$$\overset{1}{I}{}^{\alpha}=\sum_{r=0}^{1}\left\{\overset{r}{\ddot{W}}{}^{\alpha}\sum_{k=1}^{N}{}^{(k)}\overset{r+1}{M}+\overset{r}{\ddot{W}}{}^{\lambda}\sum_{k=1}^{N}\sum_{l=1}^{k-1}[{}^{(k)}\overset{r+1}{M}-{}^{(l)}Z^{+(k)}\overset{r}{M}]^{(l)}\gamma^{\alpha}_{\cdot\lambda}\right\}. \tag{17c}$$

Here

$$\overset{0}{\ddot{W}}{}^{\alpha}\equiv\ddot{v}^{\alpha}-a^{\alpha\mu}\sum_{l=1}^{k-1}{}^{(l)}Z^{+}(\ddot{\psi}_{\omega}+\ddot{v}_{3|\omega})^{(l)}\gamma^{\omega}_{\cdot\mu}\,, \tag{18a}$$

$$\overset{1}{\ddot{W}}{}^{\alpha}\equiv\ddot{\psi}^{\alpha}+a^{\alpha\mu}\sum_{l=1}^{k-1}(\ddot{\psi}_{\omega}+\ddot{v}_{3|\omega})^{(l)}\gamma^{\omega}_{\cdot\mu} \tag{18b}$$

while the mass terms are defined as:

$${}^{(k)}\overset{r}{M}=\int_{h_{(k)}}^{(k)}\rho_0(x^3)^4\,\mathrm{d}x^3,\qquad(r=0,\ldots,2)\,. \tag{19}$$

(c) The resultant surface load couples

In equations (14), $\overset{0}{p}{}^{\alpha}$, $\overset{0}{p}{}^{3}$, $\overset{1}{p}{}^{\alpha}$ denote the surface load couples (or order zero and one). From Hamilton's principle they result as:

$$\overset{0}{p}{}^{\alpha}=[\underset{\sim}{P}^{3\alpha}]^{S^+}_{S^-}\,;\qquad \overset{0}{p}{}^{3}=\left[[\underset{\sim}{P}^{33}]^{S^+}_{S_-}-[\underset{\sim}{P}^{3\alpha}]_{S^+}\sum_{l=1}^{N-1}({}^{(N)}Z^{+}-{}^{(l)}Z^{+})^{(l)}\gamma^{\beta}_{\cdot\alpha})\right]\Big|_{\beta}\,, \tag{20a}$$

$$\overset{1}{p}{}^{\alpha}=[\underset{\sim}{P}^{3\alpha}]_{S^+}\,{}^{(N)}Z^{+}-[\underset{\sim}{P}^{3\alpha}]_{S^-}\,{}^{(1)}Z^{-}+[\underset{\sim}{P}^{3\gamma}]_{S^+}\sum_{l=1}^{N-1}({}^{(N)}Z^{+}-{}^{(l)}Z^{+})^{(l)}\gamma^{\alpha}_{\cdot\gamma}\,, \tag{20b}$$

where $\underset{\sim}{P}^{3\alpha}$ and $\underset{\sim}{P}^{33}$ denote the prescribed first Piola–Kirchhoff stress tensor components.

(d) The body force terms

In addition, the gross body forces intervening in equations (14) are:

$$\overset{0}{F}{}^{\alpha}=\sum_{k=1}^{N}{}^{(k)}\overset{0}{F}{}^{\alpha}\,, \tag{21a}$$

$$\overset{0}{F}{}^{3}=\sum_{k=1}^{N}{}^{(k)}\overset{0}{F}{}^{3}-\left[\sum_{k=1}^{N}\sum_{l=1}^{k-1}({}^{(k)}\overset{1}{F}{}^{\lambda}-{}^{(l)}Z^{+(k)}\overset{0}{F}{}^{\lambda})^{(l)}\gamma^{\beta}_{\cdot\lambda}\right]\Big|_{\beta}\,, \tag{21b}$$

$$\overset{1}{F}{}^{\alpha} = \sum_{k=1}^{N} {}^{(k)}\overset{1}{F}{}^{\alpha} + \sum_{k=1}^{N}\sum_{l=1}^{k-1} ({}^{(k)}\overset{(1)}{F}{}^{\lambda} - {}^{(l)}Z^{+(k)}\overset{0}{F}{}^{\lambda})^{(l)}\gamma^{\alpha}_{\cdot\lambda} , \tag{21c}$$

where

$$^{(k)}\overset{n}{F}{}^{\alpha} = \int_{h_{(k)}} \rho_0 H^{\alpha}(x^3)^n \, \mathrm{d}x^3 \quad n = 0, 1 , \tag{22a}$$

$$^{(k)}\overset{0}{F}{}^{3} = \int_{h_{(k)}} \rho_0 H^3 \, \mathrm{d}x^3 . \tag{22b}$$

(e) Boundary conditions (BCs) on C

From the line integral arising in equation (12) we obtain the static and geometric BCs on C_f and C_v, respectively (where $C \equiv C_f \cup C_v$, $C_f \cap C_v = \emptyset$). These are:

$$\delta v_{\nu}: \quad \overset{0}{L}{}^{\alpha\beta}\nu_{\alpha}\nu_{\beta} = \overset{0}{\underset{\sim}{L}}{}^{\alpha\beta}\nu_{\alpha}\nu_{\beta} , \tag{23a}$$

$$\delta v_{\tau}: \quad \overset{0}{L}{}^{\alpha\beta}\tau_{\alpha}\nu_{\beta} = \overset{0}{\underset{\sim}{L}}{}^{\alpha\beta}\tau_{\alpha}\nu_{\beta} , \tag{23b}$$

$$\delta v_3: \quad [Q^{\beta} + \overset{0}{L}{}^{\alpha\beta}v_{3|\alpha}]\nu_{\beta} + \frac{\partial}{\partial s}(K^{\alpha\beta}\tau_{\alpha}\nu_{\beta}) = (\underset{\sim}{L}{}^{\beta} + \underset{\sim}{R}{}^{\beta})\nu_{\beta} + \frac{\partial}{\partial s}(\underset{\sim}{K}{}^{\alpha\beta}\tau_{\alpha}\nu_{\beta}] - \overset{0}{I}{}^{3\beta}\nu_{\beta} , \tag{23c}$$

$$\delta(\partial w_3/\partial\nu): \quad K^{\alpha\beta}\nu_{\alpha}\nu_{\beta} = \underset{\sim}{K}{}^{\alpha\beta}\nu_{\alpha}\nu_{\beta} , \tag{23d}$$

$$\delta\psi_{\nu}: \quad (\overset{1}{L}{}^{\alpha\beta} - K^{\alpha\beta})\nu_{\alpha}\nu_{\beta} = (\overset{1}{L}{}^{\alpha\beta} - \underset{\sim}{K}{}^{\alpha\beta})\nu_{\alpha}\nu_{\rho} , \tag{23e}$$

$$\delta\psi_{\tau}: \quad (\overset{1}{L}{}^{\alpha\beta} - K^{\alpha\beta})\tau_{\alpha}\nu_{\beta} = (\overset{1}{L}{}^{\alpha\beta} - \underset{\sim}{K}{}^{\alpha\beta})\tau_{\alpha}\nu_{\beta} . \tag{23f}$$

and at all corner points of C_f located at $s = s_i$, $i = 1, 2, \ldots$

$$\delta v_3(s_i): \quad (K^{\alpha\beta}\tau_{\alpha}\nu_{\beta})\big|_{s_i-0}^{s_i+0} = (\underset{\sim}{K}{}^{\alpha\beta}\tau_{\alpha}\nu_{\beta})\big|_{s_i-0}^{s_i+0} . \tag{23g}$$

In equations (23) $\boldsymbol{\nu}(\equiv \nu^{\alpha}\mathbf{a}_{\alpha})$ and $\boldsymbol{\tau}(\equiv \tau^{\alpha}\mathbf{a}_{\alpha})$ denote the outward normal and tangential unit vectors to C, respectively, while

$$\overset{n}{\underset{\sim}{L}}{}^{\alpha\beta} = \sum_{k=1}^{N} {}^{(k)}\overset{n}{\underset{\sim}{L}}{}^{\alpha\beta} , \quad n = 0, 1 \tag{24a}$$

$$\underset{\sim}{L}{}^{\beta} = \sum_{k=1}^{N} {}^{(k)}\underset{\sim}{L}{}^{\beta} , \tag{24b}$$

$$\underset{\sim}{K}{}^{\alpha\beta} = \sum_{k=1}^{N}\sum_{l=1}^{k-1} ({}^{(l)}Z^{+(k)}\overset{0}{\underset{\sim}{L}}{}^{\lambda\beta} - {}^{(k)}\overset{1}{\underset{\sim}{L}}{}^{\lambda\beta})^{(l)}\gamma^{\alpha}_{\cdot\lambda} \tag{24c}$$

$$\overset{0}{I}{}^{3\beta} = \sum_{r=0}^{1} \overset{r}{\ddot{W}}{}^{\lambda} \sum_{k=1}^{N}\sum_{l=1}^{k-1} [{}^{(k)}\overset{r+1}{M} - {}^{(l)}Z^{+(k)}\overset{r}{M}]^{(l)}\gamma^{\beta}_{\cdot\lambda} , \tag{24d}$$

$$R^{\beta} = \sum_{k=1}^{N} \sum_{l=1}^{k-1} ({}^{(k)}\overset{1}{F}{}^{\lambda} - {}^{(l)}Z^{+(k)}\overset{0}{F}{}^{\lambda})^{(l)}\gamma^{\beta}_{\cdot\lambda} \tag{24e}$$

with

$$\{{}^{(k)}\overset{0}{\underset{\sim}{L}}{}^{\alpha\beta}, {}^{(k)}\overset{1}{\underset{\sim}{L}}{}^{\alpha\beta}\} = \int_{h_{(k)}} {}^{(k)}\underset{\sim}{P}{}^{\beta\alpha}(1, x^3)\,\mathrm{d}x^3 \tag{24f}$$

$${}^{(k)}\overset{0}{\underset{\sim}{L}}{}^{3\beta} = \int_{h_{(k)}} {}^{(k)}\underset{\sim}{P}{}^{\beta 3}\,\mathrm{d}x^3 \,. \tag{24g}$$

The geometric BCs on C_v are given by:

$$v_{\nu} = \underset{\sim}{v}_{\nu}; \quad v_{\tau} = \underset{\sim}{v}_{\tau}; \quad v_3 = \underset{\sim}{v}_3; \quad \partial v_3/\partial\nu = \partial\underset{\sim}{v}_3/\partial\nu\,;$$
$$\psi_{\nu} = \underset{\sim}{\psi}_{\nu}; \quad \psi_{\tau} = \underset{\sim}{\psi}_{\tau}\,. \tag{25a)–(25f}$$

In equations (23) and (25), the undertilted quantities denote prescribed quantities.

Consistent with the number of six boundary conditions which have to be prescribed at each edge of the plate, the associated governing equations are of twelfth order, i.e., two orders higher than the order of the standard first order transverse shear deformation theory. This means that the fulfillment of the static conditions between the adjacent layers is paid by an increase of the order of the governing equations. This shows again that the present theory represents a refinement of the standard FSDT.

Constitutive equations

The material of each constituent layer is assumed homogeneous and anisotropic, the anisotropy being of the symmetry type with respect to the plane $x^3 = 0$. In this case the material is termed *monoclinic*. For the geometrically non-linear theory (but physically linear one), a linear relationship between the second Piola–Kirchhoff stress and Lagrangian strain tensors could be established [27]. Assuming also the existence of a non-uniform (but stationary) temperature field $T(\equiv T(x_{\alpha}, x_3)$ and postulating that the elastic properties are temperature independent, this relationship is [26]:

$${}^{(k)}S^{\alpha\beta} = {}^{(k)}\tilde{E}^{\alpha\beta\omega\rho}\,{}^{(k)}E_{\omega\rho} + {}^{(k)}\tilde{\lambda}^{\alpha\beta}T\,, \tag{26a}$$

$${}^{(k)}S^{\alpha 3} = 2\,{}^{(k)}E^{\alpha 3\omega 3}\,{}^{(k)}E_{\omega 3}\,, \tag{26b}$$

where

$$\tilde{E}^{\alpha\beta\omega\rho} \equiv E^{\alpha\beta\omega\rho} - \frac{E^{\alpha\beta 33}E^{33\omega\rho}}{E^{3333}}\,, \tag{26c}$$

and

$$\tilde{\lambda}^{\alpha\beta} = \lambda^{\alpha\beta} - \frac{E^{\alpha\beta 33}}{E^{3333}}\lambda^{33} \tag{26d}$$

denote the modified elastic and thermal expansion coefficients, respectively [26]. Substitution of equations (26) into (16) yields the constitutive equations

$$ {}^{(k)}\overset{0}{L}{}^{\alpha\beta} = {}^{(k)}_{\;0}B^{\alpha\beta\omega\rho}\, {}^{(k)}\overset{0}{e}_{\omega\rho} + {}^{(k)}_{\;1}B^{\alpha\beta\omega\rho}\, {}^{(k)}\overset{1}{e}_{\omega\rho} + {}^{(k)}_{\;0}\mathcal{T}^{\alpha\beta}\,, \tag{27a} $$

$$ {}^{(k)}\overset{1}{L}{}^{\alpha\beta} = {}^{(k)}_{\;1}B^{\alpha\beta\omega\rho}\, {}^{(k)}\overset{0}{e}_{\omega\rho} + {}^{(k)}_{\;2}B^{\alpha\beta\omega\rho}\, {}^{(k)}\overset{(1)}{e}_{\omega\rho} + {}^{(k)}_{\;1}\mathcal{T}^{\alpha\beta}\,, \tag{27b} $$

$$ {}^{(k)}\overset{0}{L}{}^{\alpha 3} = K^2\, {}^{(k)}_{\;0}B^{\alpha 3\omega 3}\, {}^{(k)}\overset{0}{e}_{\omega 3}\,. \tag{27c} $$

where

$$ \{{}^{(k)}_{\;0}B^{\alpha\beta\omega\rho},\; {}^{(k)}_{\;1}B^{\alpha\beta\omega\rho},\; {}^{(k)}_{\;2}B^{\alpha\beta\omega\rho}\} = \int_{h_{(k)}} {}^{(k)}\tilde{E}^{\alpha\beta\omega\rho}[1, x^3, (x^3)^2]\,\mathrm{d}x^3\,, \tag{27d} $$

denote the stretching, coupling bending-stretching and the bending rigidities, respectively, while

$$ \{{}^{(k)}_{\;0}\mathcal{T}^{\alpha\beta},\; {}^{(k)}_{\;1}\mathcal{T}^{\alpha\beta}\} = \int_{h_{(k)}} {}^{(k)}\tilde{\lambda}^{\alpha\beta}T[1, x^3]\,\mathrm{d}x^3 \tag{27e} $$

denote the thermal stress resultants and stress couples associated with a kth layer, respectively. In addition

$$ {}^{(k)}_{\;0}B^{\alpha 3\omega 3} = \int_{h_{(k)}} {}^{(k)}E^{\alpha 3\omega 3}\,\mathrm{d}x^3\,, \tag{28} $$

defines the transverse shear rigidities while K^2 denotes a transverse shear correction factor. As is well known, the values for $K^2 = 5/6$, $\pi^2/12$ or $2/3$ advanced in the literature by Reissner, Mindlin and Uflyand, respectively, and determined on various criteria are appropriate for a single-layered plate/shell. For a composite laminated plate K^2 becomes a solution dependent parameter. For the present case, however, as it will be shown later, there is a fixed criterium of selecting the proper value of K^2.

It may easily be shown that the following symmetry relations are valid:

$$ \begin{gathered} {}^{(k)}_{\;n}B^{\alpha\beta\omega\rho} = {}^{(k)}_{\;n}B^{\omega\rho\alpha\beta} = {}^{(k)}_{\;n}B^{\beta\alpha\omega\rho} = {}^{(k)}_{\;n}B^{\alpha\beta\rho\omega} \quad (n = 0, \ldots, 2) \\ {}^{(k)}_{\;0}B^{\alpha 3\omega 3} = {}^{(k)}_{\;0}B^{\omega 3\alpha 3} \quad \text{and} \quad {}^{(k)}_{\;n}\mathcal{T}^{\alpha\beta} = {}^{(k)}_{\;n}\mathcal{T}^{\beta\alpha} \quad (n = 0, 1)\,. \end{gathered} \tag{29} $$

In the case of symmetric laminated plates (i.e., when the plate exhibits geometric, elastic and thermal symmetry properties throughout its thickness), the reference plane could be chosen to coincide with the mid-plane of the structure. Moreover, it may easily be verified that for symmetrically laminated plates, the linearized equations and boundary conditions split exactly and entirely into two groups, belong to bending and stretching states of stress, respectively. For such a case, the governing equations associated with bending and stretching result in eighth and fourth order systems of equations, respectively.

The disparity in their order reveals that the refinements of the present theory involve the bending theory only.

Several remarks on the formulation of the geometrically non-linear theory of composite plates

In the general case of anisotropic composite plates, the problem could be formulated in terms of the five displacement functions u_α, u_3 and ψ_α. To this end, the five equations of motion (14) have to be expressed in terms of the displacement quantities by adequately using the constitutive and strain-displacement equations. However, in some problems, for example, the buckling and postbuckling ones, it is advisable to use other formulation. In the spirit of this formulation, the homogeneous counterpart of (14a), i.e.,

$$\overset{0}{L}{}^{\alpha\beta}\Big|_\beta \left[\equiv \sum_{k=1}^{N} {}^{(k)}L^{\alpha\beta}\Big|_\beta\right] = 0 \tag{30}$$

could identically be fulfilled by expressing $\overset{0}{L}{}^{\alpha\beta}$ (or ${}^{(k)}L^{\alpha\beta}$) as:

$$\overset{0}{L}{}^{\alpha\beta} = \epsilon^{\alpha\lambda}\epsilon^{\beta\omega}C|_{\lambda\mu} \quad \text{or} \; {}^{(k)}L^{\alpha\beta} = \epsilon^{\alpha\lambda}\epsilon^{\beta\omega(k)}C|_{\lambda\omega} \tag{31}$$

wherefrom it results that

$$C(x^\omega, t) = \sum_{k=1}^{N} {}^{(k)}C(x^\omega, t)\,. \tag{32}$$

In equation (31) $\epsilon^{\alpha\beta}$ denotes the 2-D permutation tensor. Being identically satisfied by (31), equations of equilibrium (14a) have to be replaced by the compatibility equations (associated with the in-plane strains) which constitute a part of the equations to be fulfilled. Elimination of v_α in equation (9a) yields the compatibility equation as:

$$\epsilon^{\alpha\pi}\epsilon^{\beta\lambda}\left[{}^{(k)}\overset{0}{e}_{\alpha\beta|\pi\lambda} + \tfrac{1}{2}u_3|_{\alpha\beta}u_3|_{\pi\lambda} + \tfrac{1}{2}\sum_{l=1}^{k-1} {}^{(l)}Z[{}^{(l)}\Omega_{\alpha|\beta\pi\lambda} + {}^{(l)}\Omega_{\beta|\alpha\pi\lambda}]\right] = 0\,, \quad k = 1, \ldots, N \tag{33}$$

On the other hand, partial inversion of constitutive equations (27a) yields:

$$\cdot\, {}^{(k)}\overset{0}{e}_{\alpha\beta} = {}^{(k)}_{\;0}S_{\alpha\beta\omega\rho}[{}^{(k)}\overset{0}{L}{}^{\omega\rho} - {}^{(k)}_{\;1}B^{\omega\rho\sigma\lambda}\,{}^{(k)}\overset{(1)}{e}_{\sigma\lambda} - {}^{(k)}_{\;0}\mathcal{T}^{\omega\rho}] \tag{34a}$$

where ${}^{(k)}_{\;0}S_{\alpha\beta\omega\rho}$ plays the role of the inverse of ${}^{(k)}_{\;0}B^{\alpha\beta\lambda\sigma}$ in the sense of

$${}^{(k)}_{\;0}S_{\alpha\beta\omega\rho}\,{}^{(k)}_{\;0}B^{\alpha\beta\lambda\sigma} = \tfrac{1}{2}(\delta^\lambda_\omega\delta^\sigma_\rho + \delta^\sigma_\omega\delta^\lambda_\rho) \tag{34b}$$

Employment of (27b,c), (34) and (31) in equations (14b,c) and (33) yields the required system of four coupled governing equations in terms of four unknown functions, C, u_3 and ψ_α. Its linearized counterpart decouples yielding an eighth order PDES in terms of u_3 and ψ_α, governing the bending and one equation of the fourth order in terms of C, governing the stretching.

Several theorems of the linearized static theory of composite laminated plates

In several earlier works it was shown that most of the general theorems of 3-D linear theory of elastostatics have analogues in the theory of plates and shells (see [26, 29] for classical and

[11, 12, 30] for refined shell theories). Within the present work it will be shown that the static counterpart of the linearized theory developed here is characterized by the same feature. To this end several energy relations associated with this theory will be displayed.

Virtual work principle (VWP)

As in the 3-D elastostatics, in this case, the VWP can be formulated in the usual way as:

$$\delta A = \delta U \tag{35}$$

where

$$\begin{aligned}\delta A = & \int_\sigma [(\overset{0}{p}{}^\alpha + \overset{0}{F}{}^\alpha)\,\delta v_\alpha + (\overset{0}{p}{}^3 + \overset{0}{F}{}^3)\,\delta v_3 + (\overset{1}{p}{}^\alpha + \overset{1}{F}{}^\alpha)\,\delta\psi_\alpha]\,\mathrm{d}\sigma \\ & + \int_{C_f} \left(\overset{0}{\underset{\sim}{L}}{}^{\alpha\beta}\nu_\alpha\,\delta v_\beta + \overset{1}{\underset{\sim}{L}}{}^{\alpha\beta}\nu_\alpha\,\delta\psi_\beta + \underset{\sim}{N}\,\delta v_3 + \underset{\sim}{M}\delta\left(\frac{\partial v_3}{\partial \nu}\right)\right)\mathrm{d}s + \sum_i [\underset{\sim}{K}{}^{\alpha\beta}\tau_\alpha\nu_\beta]|_{s_i-0}^{s_i+0}\,\delta v_3(s_i)\end{aligned} \tag{36}$$

denotes the mechanical work performed by the external and body forces and the edge loads through the virtual displacements, $\underset{\sim}{N}$ and $\underset{\sim}{M}$ are given by the right-hand sides of equations (23c,d), respectively, and

$$\delta U = \int_\sigma \sum_{k=1}^{N} \{{}^{(k)}\overset{0}{L}{}^{\alpha\beta}\delta\,{}^{(k)}\overset{0}{e}_{\alpha\beta} + {}^{(k)}\overset{1}{L}{}^{\alpha\beta}\delta\,{}^{(k)}\overset{1}{e}_{\alpha\beta} + 2\,{}^{(k)}\overset{0}{L}{}^{\alpha 3}\delta\,{}^{(k)}\overset{0}{e}_{\alpha 3}\}\,\mathrm{d}\sigma\,, \tag{37}$$

represents the internal virtual work where $\delta\,{}^{(k)}\overset{0}{e}_{\alpha\beta}$, $\delta\,{}^{(k)}\overset{1}{e}_{\alpha\beta}$, $\delta\,{}^{(k)}\overset{0}{e}_{\alpha 3}$ are related to the virtual displacements δv_α, δv_3 and $\delta\psi_\alpha$ by the linearized counterpart of equation (9). The strain energy function per unit area of the reference plane is defined as:

$$\bar{W} = \sum_{k=1}^{N} \int_{h_{(k)}} W^{(k)}\,\mathrm{d}x^3\,, \tag{38}$$

where

$$\begin{aligned}W^{(k)} = & \tfrac{1}{2}\,{}^{(k)}\tilde{E}^{\alpha\beta\omega\rho}({}^{(k)}\overset{0}{e}_{\omega\rho}\,{}^{(k)}\overset{0}{e}_{\alpha\beta} + x^3({}^{(k)}\overset{0}{e}_{\omega\rho}\,{}^{(k)}\overset{1}{e}_{\alpha\beta} + {}^{(k)}\overset{1}{e}_{\omega\rho}\,{}^{(k)}\overset{0}{e}_{\alpha\beta}) \\ & + (x^3)^2\,{}^{(k)}\overset{1}{e}_{\alpha\beta}\,{}^{(k)}\overset{1}{e}_{\omega\rho}) + 2\,{}^{(k)}E^{\alpha 3\omega 3}K^{2}\,{}^{(k)}\overset{0}{e}_{\alpha 3}\,{}^{(k)}\overset{0}{e}_{\omega 3}\,. \\ & + \tfrac{1}{2}\,{}^{(k)}\tilde{\lambda}^{\alpha\beta}({}^{(k)}\overset{0}{e}_{\alpha\beta} + x^{3}\,{}^{(k)}\overset{1}{e}_{\alpha\beta})\,T\,.\end{aligned} \tag{39}$$

Furthermore, the strain energy associated with a kth layer is

$$\begin{aligned}\bar{W}^{(k)} = & \int_{{}^{(k)}Z^-}^{{}^{(k)}Z^+} W^{(k)}\,\mathrm{d}x^3 = \tfrac{1}{2}[{}^{(k)}_{0}B^{\alpha\beta\omega\rho}\,{}^{(k)}\overset{0}{e}_{\alpha\beta}\,{}^{(k)}\overset{0}{e}_{\omega\rho} + {}^{(k)}_{1}B^{\alpha\beta\omega\rho}({}^{(k)}\overset{0}{e}_{\alpha\beta}\,{}^{(k)}\overset{1}{e}_{\omega\rho} + {}^{(k)}\overset{1}{e}_{\alpha\beta}\,{}^{(k)}\overset{0}{e}_{\omega\rho}) \\ & + {}^{(k)}_{2}B^{\alpha\beta\omega\rho}\,{}^{(k)}\overset{1}{e}_{\alpha\beta}\,{}^{(k)}\overset{1}{e}_{\omega\rho}] + 2K^2\,{}^{(k)}_{0}B^{\alpha 3\omega 3}\,{}^{(k)}\overset{0}{e}_{\alpha 3}\,{}^{(k)}\overset{0}{e}_{\omega 3} + \tfrac{1}{2}[{}^{(k)}_{0}\mathcal{T}^{\alpha\beta}\,{}^{(k)}\overset{0}{e}_{\alpha\beta} + {}^{(k)}_{1}\mathcal{T}^{\alpha\beta}\,{}^{(k)}\overset{1}{e}_{\alpha\beta}]\,.\end{aligned} \tag{40}$$

while the total strain energy stored in the structure is:

$$U = \int_\sigma \sum_{k=1}^{N} \bar{W}^{(k)}\,\mathrm{d}\sigma\,. \tag{41}$$

Theorem of work and energy (Clapeyron's theorem)

Considering in equations (40) and (41) instead of the virtual displacements δv_α, δv_3, $\delta\psi_\alpha$, the displacements corresponding to the elastic state produced by an external force system, we obtain

$$\int_\sigma [(\overset{0}{p}{}^{\alpha} + \overset{0}{F}{}^{\alpha})v_\alpha + (\overset{0}{p}{}^{3} + \overset{0}{F}{}^{3})v_3 + (\overset{1}{p}{}^{\alpha} + \overset{1}{F}{}^{\alpha})\psi_\alpha]\,\mathrm{d}\sigma$$
$$+ \int_C \left[\overset{0}{L}{}^{\alpha\beta}\nu_\alpha v_\beta + \overset{1}{L}{}^{\alpha\beta}\nu_\alpha\psi_\beta + N v_3 + M\left(\frac{\partial v_3}{\partial\nu}\right)\right]\mathrm{d}s + \sum_i [K^{\alpha\beta}\tau_\alpha\nu_\beta]\big|_{s_i-0}^{s_i+0}\, v_3(s_i) = 2U\,. \tag{42}$$

where the left-hand side of this equation expresses the work done by the external forces and prescribed edge loadings (for the sake of brevity the undertildes were suppressed).

In terms of this theorem this work is equal to twice the strain energy stored in the structure.

Betti's reciprocal theorem

According to this theorem, for the same composite plate subjected to two force and temperature field systems identified, respectively, by one and two overtildes, the work done by the first force system over the displacements due to the second one, is equal to the work done by the second force system over the displacements due to the first one. By virtue of equation (35) and of the symmetries (29), having in view that:

$$[{}_0^{(k)}B^{\alpha\beta\omega\rho}\,{}^{(k)}\overset{\tilde{0}}{e}_{\alpha\beta}\,{}^{(k)}\overset{\tilde{\tilde{0}}}{e}_{\omega\rho} + {}_1^{(k)}B^{\alpha\beta\omega\rho}({}^{(k)}\overset{\tilde{0}}{e}_{\alpha\beta}\,{}^{(k)}\overset{\tilde{\tilde{1}}}{e}_{\omega\rho} + {}^{(k)}\overset{\tilde{1}}{e}_{\alpha\beta}\,{}^{(k)}\overset{\tilde{\tilde{0}}}{e}_{\omega\rho})$$
$$+ {}_2^{(k)}B^{\alpha\beta\omega\rho}\,{}^{(k)}\overset{\tilde{1}}{e}_{\alpha\beta}\,{}^{(k)}\overset{\tilde{\tilde{1}}}{e}_{\omega\rho} + 2K^2\,{}_0^{(k)}B^{\alpha3\omega3}\,{}^{(k)}\overset{\tilde{0}}{e}_{\alpha3}\,{}^{(k)}\overset{\tilde{\tilde{0}}}{e}_{\omega3}]$$
$$= [{}_0^{(k)}B^{\alpha\beta\omega\rho}\,{}^{(k)}\overset{\tilde{\tilde{0}}}{e}_{\alpha\beta}\,{}^{(k)}\overset{\tilde{0}}{e}_{\omega\rho} + {}_1^{(k)}B^{\alpha\beta\omega\rho}({}^{(k)}\overset{\tilde{\tilde{0}}}{e}_{\alpha\beta}\,{}^{(k)}\overset{\tilde{1}}{e}_{\omega\rho} + {}^{(k)}\overset{\tilde{\tilde{1}}}{e}_{\alpha\beta}\,{}^{(k)}\overset{\tilde{0}}{e}_{\omega\rho})$$
$$+ {}_2^{(k)}B^{\alpha\beta\omega\rho}\,{}^{(k)}\overset{\tilde{\tilde{1}}}{e}_{\alpha\beta}\,{}^{(k)}\overset{\tilde{1}}{e}_{\omega\rho} + 2K^2\,{}_0^{(k)}B^{\alpha3\omega3}\,{}^{(k)}\overset{\tilde{\tilde{0}}}{e}_{\alpha3}\,{}^{(k)}\overset{\tilde{0}}{e}_{\omega3}]\,,$$
$$= ({}_0^{(k)}\tilde{\mathcal{T}}^{\alpha\beta}\,{}^{(k)}\overset{\tilde{\tilde{0}}}{e}_{\alpha\beta} + {}_1^{(k)}\tilde{\mathcal{T}}^{\alpha\beta}\overset{(\tilde{\tilde{1}})}{e}_{\alpha\beta}) - ({}_0^{(k)}\tilde{\tilde{\mathcal{T}}}^{\alpha\beta}\,{}^{(k)}\overset{\tilde{0}}{e}_{\alpha\beta} + {}_1^{(k)}\tilde{\tilde{\mathcal{T}}}^{\alpha\beta}\overset{(\tilde{1})}{e}_{\alpha\beta})\,, \tag{43}$$

in conjunction with equation (36) one obtains:

$$\int_\sigma [\overset{\tilde{\tilde{0}}}{p}{}^{\alpha}\tilde{v}_\alpha + \overset{\tilde{\tilde{0}}}{p}{}^{3}\tilde{v}_3 + \overset{\tilde{\tilde{1}}}{p}{}^{\alpha}\tilde{\psi}_\alpha]\,\mathrm{d}\sigma - \int_\sigma ({}_0\tilde{\tilde{\mathcal{T}}}^{\alpha\beta}\overset{\tilde{0}}{e}_{\alpha\beta} + {}_1\tilde{\tilde{\mathcal{T}}}^{\alpha\beta}\overset{(\tilde{1})}{e}_{\alpha\beta})\,\mathrm{d}\sigma$$
$$+ \int_C \left[\overset{\tilde{\tilde{0}}}{L}{}^{\alpha\beta}\nu_\alpha\tilde{v}_\beta + \overset{\tilde{\tilde{1}}}{L}{}^{\alpha\beta}\nu_\alpha\tilde{\psi}_\beta + \tilde{\tilde{N}}\tilde{v}_3 + \tilde{\tilde{M}}\left(\frac{\partial\tilde{v}_3}{\partial\nu}\right)\right]\mathrm{d}s + \sum_i [\tilde{\tilde{K}}{}^{\alpha\beta}\tau_\alpha\nu_\beta]_{s_i-0}^{s_i+0}\,\tilde{v}_3(s_i)$$
$$= \int_\sigma [\overset{\tilde{0}}{p}{}^{\alpha}\tilde{\tilde{v}}_\alpha + \overset{\tilde{0}}{p}{}^{3}\tilde{\tilde{v}}_3 + \overset{\tilde{1}}{p}{}^{\alpha}\tilde{\tilde{\psi}}_\alpha]\,\mathrm{d}\sigma - \int_\sigma ({}_0\tilde{\mathcal{T}}^{\alpha\beta}\overset{(\tilde{\tilde{0}})}{e}_{\alpha\beta} + {}_1\tilde{\mathcal{T}}^{\alpha\beta}\overset{(\tilde{\tilde{1}})}{e}_{\alpha\beta})\,\mathrm{d}\sigma$$

$$+ \int_C \left[\overset{\tilde{0}}{L}{}^{\alpha\beta} \nu_\alpha \tilde{\tilde{v}}_\beta + \overset{\tilde{1}}{L}{}^{\alpha\beta} \nu_\alpha \tilde{\tilde{\psi}}_\beta + \tilde{N} \tilde{\tilde{v}}_3 + \tilde{M} \frac{\partial \tilde{\tilde{v}}_3}{\partial \nu} \right] ds + \sum_i \left[\tilde{K}^{\alpha\beta} \tau_\alpha \nu_\beta \right]_{s_i - 0}^{s_i + 0} \tilde{\tilde{v}}_3(s_i) \, . \tag{44}$$

which proves the theorem. In equation (44)

$$_n\mathcal{T}^{\alpha\beta} = \sum_{k=1}^{N} {}^{(k)}_{\;n}\mathcal{T}^{\alpha\beta} \, , \quad (n = 0, 1) \, .$$

Uniqueness of solution

Let v_α, v_3, ψ_α, $\overset{0}{e}_{\alpha\beta}$, $\overset{1}{e}_{\alpha\beta}$, $\overset{0}{e}_{\alpha 3}$, $\overset{0}{L}{}^{\alpha\beta}$, $\overset{1}{L}{}^{\alpha\beta}$, $\overset{0}{L}{}^{\alpha 3}$ be a system of functions characterizing the state of stress and strain in a laminated composite anisotropic plate. These functions are determined by the solution of the linearized counterpart of equations (9), (14), and (27) for given boundary conditions (23) and (25). In order to investigate the uniqueness problem we assume that for given boundary conditions there exist two sets of solutions identified by $(^{(A)})$ and $(^{(B)})$, respectively. By virtue of the linear character of field equations, the system of functions characterizing the difference solution (identified by the superscript $(^{d})$) satisfies the constitutive equations under the form where the thermal terms disappear, the geometrical equations, the homogeneous equations of equilibrium, as well as the homogeneous boundary conditions. Transposing (42) for the difference solution we obtain that $\overset{(d)}{U} = 0$, implying that $(\overset{0}{e}_{\alpha\beta}, \overset{1}{e}_{\alpha\beta}, \overset{0}{e}_{\alpha 3})^d = 0$.

It may be concluded from this that

$$(\overset{0}{e}_{\alpha\beta}, \overset{1}{e}_{\alpha\beta}, \overset{0}{e}_{\alpha 3})^{(A)} = (\overset{0}{e}_{\alpha\beta}, \overset{1}{e}_{\alpha\beta}, \overset{0}{e}_{\alpha 3})^{(B)}$$

and on the basis of (27) it results

$$(\overset{0}{L}{}^{\alpha\rho}, \overset{1}{L}{}^{\alpha\rho}, \overset{0}{L}{}^{\alpha 3})^{(A)} = (\overset{0}{L}{}^{\alpha\rho}, \overset{1}{L}{}^{\alpha\rho}, \overset{0}{L}{}^{\alpha 3})^{(B)} \, ,$$

that both solutions give the same state of stresses and strain. In the case of the boundary conditions expressed in terms of stress resultants, the previous result does not imply that $[v_\alpha, v_3, \psi_\alpha]^d \equiv 0$. In this case the solutions $[v_\alpha, v_3, \psi_\alpha]^{(A)}$ and $[v_\alpha, v_3, \psi_\alpha]^{(B)}$ may differ by a rigid-body displacement which has no effect on the state of stress or strain of the composite plate.

Minimum potential (MPE) and complementary energy (MCE) principles

MPE states that of all the *kinematical admissible* displacement fields v'_α, v'_3, ψ'_α, the *actual* one yields the absolute minimum of the potential energy functional. To this end we define the functional

$$\begin{aligned} \mathscr{E} = {} & \int_\sigma \sum_{k=1}^{N} \bar{W}^{(k)} \, d\sigma - \int_\sigma [(\overset{0}{p}{}^\alpha + \overset{0}{F}{}^\alpha) v_\alpha + (\overset{0}{p}{}^3 + \overset{0}{F}{}^3) v_3 \\ & + (\overset{1}{p}{}^\alpha + \overset{1}{F}{}^\alpha) \psi_\alpha] \, d\sigma - \int_C \left(\overset{0}{L}{}^{\alpha\beta} \nu_\alpha v_\beta + \overset{1}{L}{}^{\alpha\beta} \nu_\alpha \psi_\beta + N v_3 + M \left(\frac{\partial v_3}{\partial \nu} \right) \right) ds \\ & - \sum_i \left[K^{\alpha\beta} \tau_\alpha \nu_\beta \right]_{s_i - 0}^{s_i + 0} v_3(s_i) \end{aligned} \tag{45}$$

where $\bar{W}^{(k)}$ is given by (40).

Let us express the *kinematical admissible* displacement field $(v'_\alpha, v'_3, \psi'_\alpha)$ in terms of its *actual* counterpart $(v_\alpha, v_3, \psi_\alpha)$ as

$$v'_\alpha = v_\alpha + \delta v_\alpha \; ; \quad v'_3 = v_3 + \delta v_3 \; ; \quad \psi'_\alpha = \psi_\alpha + \delta\psi_\alpha \tag{46}$$

where δv_α, δv_3 and $\delta\psi_\alpha$ are zero on C_v. Then, for the difference between the potential energies $\mathscr{E}'$ and $\mathscr{E}$ corresponding to the *kinematically admissible* and the *actual displacement* fields, respectively, we have:

$$\begin{aligned}\mathscr{E}' - \mathscr{E} = &\int_\sigma \sum_{k=1}^{N} [{}^{(k)}\bar{W}'({}^{(k)}\overset{0}{e}_{\alpha\rho} + \delta\,{}^{(k)}\overset{0}{e}_{\alpha\rho} \; ; \quad {}^{(k)}\overset{1}{e}_{\alpha\rho} + \delta\,{}^{(k)}\overset{1}{e}_{\alpha\rho} \; ; \quad {}^{(k)}\overset{0}{e}_{\alpha 3} + \delta\,{}^{(k)}\overset{0}{e}_{\alpha 3}) \\ &- {}^{(k)}\bar{W}({}^{(k)}\overset{0}{e}_{\alpha\rho}, {}^{(k)}\overset{1}{e}_{\alpha\rho}, {}^{(k)}\overset{0}{e}_{\alpha 3})]\, d\sigma \\ &- \int_{C_f} \left(\overset{0}{L}{}^{\alpha\beta}\nu_\alpha\, \delta v_\beta + \overset{1}{L}{}^{\alpha\beta}\nu_\alpha\, \delta\psi_\beta + N\,\delta v_3 + M\delta\left(\frac{\partial v_3}{\partial \nu}\right)\right) ds \\ &- \sum_i \,[K^{\alpha\beta}\tau_\alpha \nu_\beta]_{s_i-0}^{s_i+0}\, \delta v_3(s_i)\,.\end{aligned} \tag{47}$$

Expanding $\bar{W}'$ into a power series we have

$$\bar{W}' = \bar{W} + \delta\bar{W} + \frac{1}{2}\delta^2\bar{W}\,, \tag{48}$$

where

$$\delta\bar{W}^{(k)} \equiv \frac{\delta\bar{W}^{(k)}}{\delta\,{}^{(k)}\overset{0}{e}_{\alpha\beta}}\,\delta\,{}^{(k)}\overset{0}{e}_{\alpha\beta} + \frac{\delta\bar{W}^{(k)}}{\delta\,{}^{(k)}\overset{1}{e}_{\alpha\beta}}\,\delta\,{}^{(k)}\overset{1}{e}_{\alpha\beta} + \frac{\delta\bar{W}^{(k)}}{\delta\,{}^{(k)}\overset{0}{e}_{\alpha 3}}\,\delta\,{}^{(k)}\overset{0}{e}_{\alpha 3}\,, \tag{49}$$

while

$$\begin{aligned}\tfrac{1}{2}\delta^2\bar{W}^{(k)} = &\tfrac{1}{2}[{}^{(k)}_{\,0}B^{\alpha\beta\omega\rho}\delta\,{}^{(k)}\overset{0}{e}_{\alpha\beta}\delta\,{}^{(k)}\overset{0}{e}_{\omega\rho} + {}^{(k)}_{\,1}B^{\alpha\beta\omega\rho}(\delta\,{}^{(k)}\overset{0}{e}_{\alpha\beta}\delta\,{}^{(k)}\overset{1}{e}_{\omega\rho} \\ &+ \delta\,{}^{(k)}\overset{0}{e}_{\omega\rho}\delta\,{}^{(k)}\overset{1}{e}_{\alpha\beta}) + {}^{(k)}_{\,2}B^{\alpha\beta\omega\rho}\delta\,{}^{(k)}\overset{1}{e}_{\alpha\beta}\delta\,{}^{(k)}\overset{1}{e}_{\omega\rho} + 2K^2\,{}^{(k)}_{\,0}B^{\alpha 3\omega 3}\delta\,{}^{(k)}\overset{0}{e}_{\alpha 3}\delta\,{}^{(k)}\overset{0}{e}_{\omega 3}] \\ \equiv\; &\bar{W}^{(k)}(\delta\,{}^{(k)}\overset{0}{e}_{\alpha\beta}, \delta\,{}^{(k)}\overset{1}{e}_{\alpha\beta}, \delta\,{}^{(k)}\overset{0}{e}_{\alpha 3})\,.\end{aligned} \tag{50}$$

Invoking the virtual work principle, we obtain

$$\mathscr{E}' - \mathscr{E} = \int_\sigma \sum_{k=1}^{N} (\delta\,{}^{(k)}\overset{0}{e}_{\alpha\beta}, \delta\,{}^{(k)}\overset{1}{e}_{\alpha\beta}, \delta\,{}^{(k)}\overset{0}{e}_{\alpha 3})\, d\sigma\,. \tag{51}$$

If $\bar{W}^{(k)}(\delta\,{}^{(k)}\overset{0}{e}_{\alpha\rho}, \delta\,{}^{(k)}\overset{1}{e}_{\alpha\rho}, \delta\,{}^{(k)}\overset{0}{e}_{\alpha 3})$ is positive definite, it results that

$$\mathscr{E}' - \mathscr{E} \geqslant 0 \tag{52}$$

for $(\delta\,{}^{(k)}\overset{0}{e}_{\alpha\rho}, \delta\,{}^{(k)}\overset{1}{e}_{\alpha\rho}, \delta\,{}^{(k)}\overset{0}{e}_{\alpha 3}) \neq 0$, and therefore $\mathscr{E}$ is a minimum.

Minimum complementary energy principle

The MCE principle states that of all *statistically admissible* stress fields (i.e., $\overset{0}{L}{}^{\alpha\rho}$, $\overset{1}{L}{}^{\alpha\rho}$ and $\overset{0}{L}{}^{\alpha 3}$), the *actual* one renders a minimum of the complementary energy functional $\mathscr{E}_C$, defined as:

$$\mathscr{E}_C = \int_\sigma \sum_{k=1}^{N} \bar{\Sigma}^{(k)}\, \mathrm{d}\sigma - \int_{C_v} \left[\overset{0}{L}{}^{\alpha\beta} \nu_\alpha v_\beta + \overset{1}{L}{}^{\alpha\beta} \nu_\alpha \psi_\beta + N v_3 + M\left(\frac{\partial v_3}{\partial \nu}\right) \right] \mathrm{d}s \tag{53}$$

where the complementary energy function $\bar{\Sigma}$ is related to $\bar{W}$ through a Legendre transformation, as:

$$\begin{aligned} \bar{\Sigma}^{(k)}({}^{(k)}\overset{0}{L}{}^{\alpha\rho}, {}^{(k)}\overset{1}{L}{}^{\alpha\rho}, {}^{(k)}\overset{0}{L}{}^{\alpha 3}) &= {}^{(k)}\overset{0}{L}{}^{\alpha\rho}\, {}^{(k)}\overset{0}{e}_{\alpha\rho} \\ &+ {}^{(k)}\overset{1}{L}{}^{\alpha\rho}\, {}^{(k)}\overset{1}{e}_{\alpha\rho} + {}^{(k)}\overset{0}{L}{}^{\alpha 3}\, {}^{(k)}\overset{0}{e}_{\alpha 3} - {}^{(k)}\bar{W}({}^{(k)}\overset{0}{e}_{\alpha\beta}, {}^{(k)}\overset{1}{e}_{\alpha\beta}, {}^{(k)}\overset{0}{e}_{\alpha 3}) \end{aligned} \tag{54}$$

As in the case of MPE principle (see also [24]), it could be shown that (53) assumes its absolute minimum for the *actual* solution and in addition that $(\mathscr{E})_{\min} = -(\mathscr{E}_C)_{\min}$. We mention in passing that these energetic principles constitute as basis, among others, of many approximate numerical solution schemes, the most popular one being the finite-element method.

Discussion

The previous developments concern the substantiation of a shear deformable theory of anisotropic composite laminated plates based upon the fulfillment of both geometrical and statical continuity conditions between the adjacent layers. It incorporates dynamic and thermal effects as well as the geometric non-linearities. Similarly to, the first order transverse shear deformation theory (FSDT) of unsymmetric laminated plates, this theory involves determination of five independent unknown functions, only. However, in contrast to the standard FSDT resulting in a tenth order governing equation system, in this case the order of governing equations is twelve, which implies prescription of six boundary conditions at each edge. As it results from the paper, the incorporation of the extra-refinement consisting of the fulfillment of the static continuity conditions at the layer interfaces becomes necessary in the case of advanced composite material structures when drastic variations in the transverse shear properties are experienced from layer to layer. In contrast to other available approaches, in the present one there is a possibility to evaluate, depending on the values of ${}^{(k)}\gamma^{\omega}_{\cdot\alpha}$, if there is a real necessity to introduce this refinement (and so to increase the order of the governing equations), or to use the standard FSDT.

In any case, for hybrid composite structures whose material layers may feature drastic variations in the transverse shear moduli and for sandwich structures constituted by thin layers (for which, theoretically, the transverse shear rigidity is infinite), separated by thick layers (which are characterized by low rigidities in transverse shear), the use of such a theory is essential.

After the deduction of the basic equations and boundary conditions, it was shown that several theorems of the linear theory of 3-D elastostatics find their analogues in the theory of composite plates developed in this paper. Related to the necessity of prescribing the global transverse shear correction factor, the following comments are in order. As is well known, in the theory of laminated composite plates (and shells) transverse shear correction factors are lamination [31, 32] and temperature [33] dependent quantities. It is also known [34] that for a single-layered plate (even when a temperature field is present [30]) $K^2 = 5/6$ is an exact shear-correction factor. In the present case, due to the fulfillment of the continuity of transverse shear stresses at the layer interfaces they result constant throughout the entire laminated thickness and, consequently, likewise in the case of a single-layered plate, $K^2 = 5/6$ could be used as an exact shear correction factor.

Due to the expected accuracy of inter- and intra-laminae prediction response characteristic, this theory could be successfully applied, for example, in the assessment of the damage response or of the failure predictions of postbuckled composite plates. Certainly the theory of plates developed here could be further refined by fulfilling the static conditions on the top and bottom faces of the plate and/or by introducing the effect of delaminations. While the former refinement as revealed in papers [32, 33] does not result in an increase of the order of government equations but only in their intricacy, the implementation of the latter one will yields extra-complexities in the analytical treatment.

Last but not least, the numerical results and the comparison of the predictions based on such a structural model with their counterparts based on the 3-D elasticity theory reveal the excellent performances of this model. It is hoped that the present developments as well as the ones done in [35–39] will enable one further applications in which their features would be essential for a reliable prediction of the failure and damage response of advanced composite structures and of a number of local phenomena related to their behavior.

Nomenclature

$\mathbf{a}_i, \mathbf{a}^i$	covariant and contravariant base vectors of the reference plane
a_{ij}, a^{ij}	covariant and contravariant metric tensor components of the reference plane
${}^{(k)}_{\ n}B^{\alpha\beta\omega\rho}, {}^{(k)}_{\ 0}B^{\alpha 3\omega 3}$	rigidities of the kth layer (of the nth and 0th order, respectively)
E_{ij}	Green–Lagrange strain tensor components
$\overset{n}{e}_{ij}$	2-D strain measures obtained as nth term of the expansion of E_{ij}
$E^{ijkl}, \tilde{E}^{\alpha\beta\omega\rho}$	tensors of elastic moduli
$\overset{n}{F}{}^{\alpha}, \overset{n}{F}{}^{3}$	2-D nth order gross body couples
${}^{(k)}\overset{n}{F}{}^{\alpha}, {}^{(k)}\overset{n}{F}{}^{3}$	nth order body couples of the kth layer
F_{ijkl}	elastic compliance tensor
$\mathbf{g}_i, \mathbf{g}^i$	3-D covariant and contravariant base vector
g_{ij}, g^{ij}	3-D covariant and contravariant metric tensor components
$h, h_{(k)}$	total plate thickness; thickness of the kth lamina
$\mathbf{H}, H^i$	body force vector and its components
$\overset{n}{I}{}^{\alpha}, \overset{n}{I}{}^{3}$	2-D nth order gross inertia couples
K	kinetic energy
K^2	transverse shear correction factor (equation (27c))
$\overset{n}{L}{}^{\alpha\beta}$	2-D nth order gross stress resultants
${}^{(k)}\overset{n}{L}{}^{\alpha\beta}, {}^{(k)}\overset{n}{L}{}^{\alpha\beta}$	2-D nth order stress results of the kth layer
${}^{(k)}\overset{n}{M}$	nth order mass term of the kth layer
$\mathbf{n}$	unit normal vector to the reference plane
$\overset{n}{p}{}^{\alpha}, \overset{n}{p}{}^{3}$	2-D nth order surface load couples
p^{ij}	first Piola–Kirchhoff stress tensor
R^B	equation (24e)
$\underset{\sim}{S}{}^i$	prescribed stress vector components

S^{ij}	second Piola–Kirchhoff stress tensor
t	time
$T(x^{\omega}, x^3)$	3-D temperature field
${}^{(k)}_{n}\mathcal{T}^{\alpha\rho}$	2-D thermal stress resultants
$\mathbf{V}, V_i$	displacement vector and its components in 3-D space
${}^{(k)}V_i$	displacement components of the kth layer
$\underset{n}{v}_i$	displacement components of the reference plane
$\ddot{W}$	nth order acceleration quantity (equation (18))
x^{α}, x^3	convective surface and normal coordinates
Y	Heaviside function
${}^{(k)}Z, {}^{(k)}Z^+, {}^{(k)}Z^-$	values of x^3 at the mid-plane, at the upper and bottom faces of the kth layer, respectively.
${}^{(k)}\gamma^{\lambda}_{\cdot\alpha}$	measure of the difference of transverse shear moduli between the $(k+1)$th and kth layer (equation (11))
$\lambda^{\alpha\beta}, \tilde{\lambda}^{\alpha\beta}$	thermal expansion coefficients
$\boldsymbol{\nu}, \nu^{\alpha}$	unit outward normal vector to the boundary and its components
${}^{(k)}\Omega_{\alpha}$	2-D functions intervening in the displacement representation (equation (5))
ρ	mass density
$\boldsymbol{\tau}, \tau^{\alpha}$	unit tangent vector of the boundary and its components
ψ_{α}	rotation angles of the normal to the reference plane

Symbols

$A_{,\alpha}$	partial derivative of A with respect to x^{α}
$A\|_{\alpha}$	covariant derivative of A with respect to x^{α}
δA	variation of A
$\underset{\sim}{A}$	prescribed quantity A

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